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The nonlinear excitation of synchronous edge waves by a monochromatic wave normally approaching a plane beach

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The possible excitation of synchronous edge waves by a monocromatic wave normally approaching a plane beach is studied. Use is made of the full three-dimensional water wave theory but only beach angles $\beta = \pi/(2N)$, where N is an integer, are considered. A weakly nonlinear stability analysis is used to investigate the interaction of subharmonic, synchronous and 3/2-frequency edge waves with the incoming wave field. It is shown that values of β exist for which an energy transfer from the incoming wave to the synchronous edge waves takes place through the intermediary of the subharmonic components.

1. Introduction

Several different mechanisms for edge wave generation on sloping beaches are possible. These trapped waves can be excited by external forces such as wind stresses acting directly above the water surface or uneven pressure distributions related to storms travelling parallel to the coast or by transient incident waves.

Edge waves may also be excited by a normally incident monochromatic wave through an instability mechanism. Indeed Guza & Davis (1974), by using the shallow water approximation, showed that a wave train of angular frequency ω^*, which is normally incident on a straight beach of constant slope β, may transfer energy to small disturbances in the form of edge waves subharmonic with respect to the incoming wave, i.e. characterized by angular frequency $\omega^*/2$. Nonlinear effects were investigated and an equilibrium state was predicted by means of a weakly nonlinear stability theory by Guza & Bowen (1976). Later Minzoni & Whitham (1977) analysed the same instability problem using the full three-dimensional water wave theory. The above work shows that in this case the non-uniformity of the shallow water approximation far from the shore described by Whitham (1976) and Minzoni (1976) is mild and does not affect the main results for small beach angles. More recently Miles (1990) using the shallow water equations has tackled the problem for a beach profile that descends smoothly from a shoreline depth of zero and slope β to an offshore small depth h_∞, thus avoiding the inconsistency of the model by Guza & Davis (1974) far from the shoreline.

While the presence of subharmonic edge waves in the nearshore region is thus explained, much less is known on edge waves which are characterized by the same angular frequency as that of the incoming wave field (synchronous edge waves). The presence of synchronous edge waves was observed experimentally by Galvin (1965) and Guza & Inman (1975) among others. Rockliff (1978) showed that synchronous edge waves can be excited by a mechanism similar to that described by Guza & Davis (1974). However her approach takes into account steady and ultraharmonic components of the incoming wave field. It turns out that the amplification rate of synchronous edge waves and their equilibrium amplitude are an order of magnitude smaller than those characterizing the subharmonic edge waves. Indeed if a denotes a small parameter related to the steepness of the incoming wave, the growth of subharmonic edge waves takes place on the slow time scale at^*, while synchronous edge waves are shown by Rockliff (1978) to grow with a rate of order a^2t^*. Moreover the final equilibrium amplitudes are $O(a^{1/2})$ and $O(a)$ for subharmonic and synchronous edge waves respectively.

In the present work an alternative mechanism for the excitation of synchronous edge waves is described which takes into account, by means of a weakly nonlinear analysis, not only the interaction of free modes with the incoming wave but also nonlinear interactions between different free modes. Use is made of the full three-dimensional water wave theory. It is found that the nonlinear effects may transfer energy from the subharmonic to the synchronous modes when suitable transverse wavelengths are considered. This energy transfer can cause the growth of both the synchronous and subharmonic modes. The possibility of subharmonic external resonance coupled with synchronous internal resonance has been already analysed in different contexts and in particular for the so-called 'Faraday resonance' problem. For a review on this topic see Miles & Henderson (1990). The dispersion relation of the free edge wave modes shows that a subharmonic–synchronous coupling is possible for sufficiently small values of the beach slope β. Particular values of β might also lead to the excitation of modes characterized by an angular frequency which is 3/2 times that of the incoming wave and possibly of higher frequency modes. In this case a more complicated system of coupled amplitude equations would be obtained. Here only the coupling among edge waves characterized by frequencies $\omega^*/2$, ω^* and $3\omega^*/2$ is considered.

The results show that values of the beach slope β exist such that synchronous edge waves can grow with an amplification rate of the same order of magnitude as that of subharmonic modes. Moreover the results indicate that values of the parameters exist such that the amplitudes of both the subharmonic and synchronous modes tend to become much larger than that of the incoming wave. In such cases an equilibrium configuration may be found by resorting to a full nonlinear approach.

The problem is of practical interest due to the role that synchronous edge waves can play on beach cusp formation. Indeed, although there is still some controversy as to the cause of beach cusp formation (Werner & Fink 1993), it seems reasonable to relate the appearance of beach cusps to the presence of steady currents periodic in the longshore direction. The interaction between synchronous edge waves and the incoming wave provides the simplest explanation for the generation of steady recirculating cells periodic in the longshore direction which in turn may modify bottom topography and the shoreline configuration.

The procedure used in the rest of the paper is as follows. In the next section we formulate the problem and in §3 we present the linear solution for the incoming wave field. The free edge wave modes and their interaction with the incoming wave, taking into account nonlinear effects, are considered in §4 where amplitude equations for the time development of a particular set of resonating edge waves are obtained. Finally, some results are presented and discussed in §5.

2. Formulation of the problem

Let us consider a straight and infinitely long beach, with the bottom sloping downward offshore at a constant rate $\tan\beta$. The special values of $\beta = \pi/(2N)$, N being a positive integer, will be considered in the following in order to find more explicit results. Orthogonal Cartesian coordinates are introduced with the x^*-axis lying on the still water surface and being directed offshore, the y^*-axis coincident with the coastline and the z^*-axis pointing upward. Let us then consider a normally incident monochromatic wave of angular frequency ω^* which is perfectly reflected by the beach.

If the flow induced by the wave is supposed to be irrotational, the dimensionless problem for the velocity potential ϕ and the free surface displacement η is posed by the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{2.1}$$

and the following boundary conditions:

$$\sin\beta\frac{\partial\phi}{\partial x}+\cos\beta\frac{\partial\phi}{\partial z}=0 \qquad \text{for } z=-x\tan\beta, \tag{2.2}$$

$$\frac{\partial\phi}{\partial t}+\eta+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]=0 \qquad \text{for } z=\eta, \tag{2.3}$$

$$\frac{\partial\eta}{\partial t}+\left[\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y}\right]-\frac{\partial\phi}{\partial z}=0 \qquad \text{for } z=\eta, \tag{2.4}$$

where $g/(\omega^*)^2$ is used as a length scale, $1/\omega^*$ as a time scale and the quantity g^2/ω^{*3} as scale for the velocity potential. In the above definitions g is acceleration due to gravity.

3. The basic wave field

If the steepness of the incoming wave is assumed to be much smaller than one, the velocity potential associated with the standing wave generated by the superposition of the incoming and reflected waves can be expanded in terms of the small parameter $a = a^*\omega^{*2}/g$, if a^* denotes the amplitude of the steady wave at the shoreline:

$$\phi = a\phi_{00} + a^2\phi_{01} + O(a^3). \tag{3.1}$$

Since g/ω^{*2} is the order of magnitude of the wavelength of the approaching wave (indeed ω^{*2}/g is its wavenumber in the deep water region) and a^* is proportional to its amplitude (see the paragraph after relationship (3.5)), the parameter a is related to the steepnes of the incoming wave.

The solution of the linearized problem reads (Stoker 1957, pp. 77–84)

$$\phi_{00}=\left[\sum_{m=1}^{N}\frac{\mathscr{A}_m}{4N^{1/2}}\exp[\mathscr{B}_m(x+\mathrm{i}z)]+\text{c.c.}\right]\mathrm{e}^{\mathrm{i}t}+\text{c.c.} \tag{3.2}$$

where

$$\mathscr{A}_m=\exp\left[\mathrm{i}\pi\left(\frac{N+1}{4}-\frac{m}{2}\right)\right]\prod_{\ell=1}^{m-1}\cot\frac{\ell\pi}{2N}, \tag{3.3}$$

$$\mathscr{A}_1 = \bar{\mathscr{A}}_N, \tag{3.4}$$

$$\mathscr{B}_m = \exp\left[\mathrm{i}\pi\left(\frac{m}{N}+\frac{1}{2}\right)\right], \tag{3.5}$$

and c.c. denotes the complex conjugate of the fore-going terms and an overbar over a quantity denotes its complex conjugate.

Notice that the solution for the velocity potential described in Stoker (1957) is defined but for the presence of an arbitrary multiplying constant the value of which can be determined by fixing the amplitude of the surface oscillations which here is a^* at x^* equal to zero. The relationship between a^* and the amplitude α^* of the incoming wave can be found by looking at the behaviour of ϕ_{00} for $x \to \infty$. Indeed ϕ_{00} far away from the shoreline tends to become the velocity potential of a standing wave of amplitude $a^*/N^{1/2}$, hence α^* turns out to be $a^*/(2N^{1/2})$.

4. The time development of edge waves

Let us now consider a perturbation of small amplitude ϵ, superimposed on the basic wave field and decaying for $x \to \infty$.

If ϵ is assumed to be much smaller than one, the following two-parameter expansion can be used to express the perturbed velocity potential:

$$\phi = a\phi_{00} + a^2\phi_{01} + O(a^3) + \epsilon\left[\phi_{10} + a\phi_{11} + O(a^2)\right] + \epsilon^2\left[\phi_{20} + a\phi_{21} + O(a^2)\right] + O(\epsilon^3). \tag{4.1}$$

4.1. *Linear analysis*

By substituting (4.1) into (2.1)–(2.4) and considering terms of $O(\epsilon)$, a problem for ϕ_{10} is obtained which of course turns out to be a linear eigenvalue problem. It is worth pointing out that ϕ_{10} describes the time development and spatial distribution of the leading-order term of the velocity potential associated with the perturbation. A normal mode analysis in the longshore direction can be applied and the generic component of the initial perturbation can be found (Ursell 1952):

$$\begin{aligned}\hat{\phi}_{10}(x,y,z;k,n) &= \exp[\mathrm{i}(ky+\sigma t)]\Bigg\{\exp[-k(x\cos\beta - z\sin\beta)] \\ &\quad + \sum_{m=1}^{n}\mathscr{C}_{mn}\{\exp[-k(x\cos((2m-1)\beta) + z\sin((2m-1)\beta))] \\ &\quad + \exp[-k(x\cos((2m+1)\beta) - z\sin((2m+1)\beta))]\}\Bigg\} \\ &= \exp[\mathrm{i}(ky+\sigma t)]\,\psi(x,z)\end{aligned} \tag{4.2}$$

where

$$\mathscr{C}_{mn} = (-1)^m \prod_{\ell=1}^{m} \frac{\tan(n-\ell+1)\beta}{\tan(n+\ell)\beta}, \tag{4.3}$$

The longshore wavenumber k is related to the angular frequency σ by the dispersion relation (eigenrelation)

$$\sigma^2 = k\sin(2n+1)\beta, \qquad n = 1,2,\ldots,\mathscr{N} \tag{4.4}$$

where, for a fixed slope β, the number $\mathcal{N}$ of modes is the greatest integer contained in $[(1/2)+(\pi/4\beta)]$.

The values of σ turn out to be real showing that linear edge waves neither amplify nor decay but simply propagate along the coast, i.e. they are marginally stable. A growth of the edge waves may take place as described in the following if the interaction with the incoming wave is taken into account. Equation (4.4) shows that the Stokes edge wave appearing for $n=0$ in (4.4) (Stokes 1846) is not the only discrete mode, but the first of a sequence. Hence the complete solution for ϕ_{10} is given by

$$\phi_{10}(x,y,z,t)=\int_{-\infty}^{+\infty}\sum_{n=0}^{\mathcal{N}}A(k,n)\hat{\phi}_{10}(x,y,z,t;k,n)\mathrm{d}k \tag{4.5}$$

where A is an amplitude function.

4.2. *Nonlinear interactions*

When the expansion (4.1) is substituted into the problem (2.1)–(2.4), the nonlinear couplings originate different quadratic terms which can be denoted by their possible combinations: $a^2(\phi_{00},\phi_{00})$, $\epsilon a(\phi_{10},\phi_{00})$, $\epsilon^2(\phi_{10},\phi_{10})$ plus terms of higher order which are also generated by cubic interactions. In non-resonant cases a solution of the problems for $\phi_{01},\phi_{11},\phi_{20}$ forced by the above terms can be easily found. In particular the interaction of the basic wave field with itself ($a^2(\phi_{00},\phi_{00})$) gives rise to a component of the velocity potential of angular frequency $2\omega^*$ and to a steady part. The term $\epsilon a(\phi_{10},\phi_{00})$ produces a slight modification of the original perturbation. Finally, the term $\epsilon^2(\phi_{10},\phi_{10})$ originates different effects and in particular an outgoing progressive wave of frequency 2σ: therefore the perturbation leaks energy to the far field at order ϵ^2 (Guza & Bowen 1976; Minzoni & Whitham 1977).

But in the special cases in which the nonlinear forcings $\epsilon a(\phi_{10},\phi_{00})$, $\epsilon^2(\phi_{10},\phi_{10})$ give rise to terms characterized by an angular frequency and a longshore wavenumber satisfying (4.4) resonance occurs. Many resonanting cases exist.

As discussed by Guza & Davis (1974) and Minzoni & Whitham (1977) the term $\epsilon a(\phi_{10},\phi_{00})$ gives rise to resonance in particular when subharmonic edge waves are considered, i.e. when $\sigma=1/2$. In this case an energy transfer from the basic wave to the edge waves takes place which induces an exponential growth of edge wave amplitude on a temporal scale equal to at^*. As discussed by Minzoni & Whitham (1977) when the edge wave amplitude increases, the cubic terms become important and limit the final amplitude. An order of magnitude argument leads to the conclusion that the final equilibrium amplitude is characterized by an order of magnitude equal to $a^{1/2}$.

However, these findings rest on the assumption that no other resonant interaction takes place. When the interaction among subharmonic, synchronous and ultraharmonic components of the perturbation is taken into account, a different mechanism may take place which transfers energy from the subharmonic modes to the synchronous ones and eventually to ultraharmonic modes. Indeed the term $\epsilon^2(\phi_{10},\phi_{10})$ may also give rise to resonance at least for particular values of β.

In order to illustrate resonance conditions let us first consider two subharmonic modes propagating in opposite directions and characterized by a wavenumber we denote k_1. The dispersion relation (4.4) shows that for suitable values of β, a pair of synchronous modes propagating in opposite directions exists which is characterized by a transverse wavenumber k_2 which differs from $2k_1$ by a small amount. In other words for particular values of β it is feasible to consider synchronous edge waves characterized by a longshore wavenumber

$$k_2 = 2k_1 + a\mu_2. \tag{4.6}$$

The term $a\mu_2$, with μ_2 a parameter of order one, takes into account the small mismatch between k_2 and $2k_1$. It can be easily appreciated that the interaction among the above-described subharmonic and synchronous modes gives rise to resonance. Indeed the interaction of the subharmonic mode propagating in the positive/negative longshore direction with itself gives rise to the synchronous mode propagating in the same direction, while the interaction of a synchronous mode with a subharmonic one propagating in the same direction gives rise to the subharmonic component itself. The possibility of subharmonic–synchronous internal resonance (Simmons 1969) has been studied in the past in different contexts (see for example Miles & Henderson 1990).

In general it is possible to single out values of β such that pairs of edge waves appear the angular frequency of which is equal to $\pm m/2$, with m an integer larger than 1, and their transverse wavenumbers k_m are close to mk_1, i.e for these particular values of β it can be assumed

$$k_m = mk_1 + a\mu_m, \qquad m = 2, 3, 4, ...M \tag{4.7}$$

where M depends on a and β. As previously pointed out, it can be recognized that the nonlinear interaction of such free modes leads to resonance. For example if $M = 3$, the interaction of the subharmonic mode propagating in the positive/negative direction with the synchronous mode propagating in the same direction gives rise to the edge waves characterized by the frequency equal to 3/2. Moreover, as before, the interaction of the subharmonic modes with themselves originates synchronous edge waves and so on. A more detailed discussion of resonance is given, for example, in Miles & Henderson (1990). Since for particular values of β the condition (4.7) may be satisfied by a second pair of edge waves with the same angular frequency but with a different mode numbers, the wavenumbers of the two pairs will be denoted by $k_m^{(1)}$ and $k_m^{(2)}$ respectively while $a\mu_m^{(1)}$ and $a\mu_m^{(2)}$ are their distances from mk_1. By considering smaller values of β and allowing fairly large values of a more pairs of edge waves satisfying (4.7) may be found. However, in the present work attention will be focused on values of N such that only two pairs are resonating.

Of course the internal resonances just described take place at order ϵ^2 while the external resonance between the subharmonic edge waves and the basic wave field is present at order ϵa. Hence when an infinitesimal perturbation is considered only the external resonance is effective. However, the latter resonance leads to a growth of the perturbation and when it becomes of $O(a)$, i.e. before reaching its equilibrium amplitude which is of order $a^{1/2}$, the external resonance couples with the internal one. This coupling may limit the growth of the subharmonic components and trigger that of the synchronous ones, a mechanism which could not be described by the scheme of Minzoni & Whitham (1977).

The final aim of the present work is to show that for particular values of the beach slope and of the incoming wave amplitude, the growth of synchronous modes can be triggered by their interaction with subharmonic edge waves. Hence in order to keep the analysis as simple as possible, we would like to neglect the possible interactions of the subharmonic and synchronous edge waves with modes characterized by an angular frequency equal to $\pm m/2$, with m equal to $3, 4, ...M$, and a longshore wavenumber described by (4.7). However, starting from $\beta = \pi/4$ and considering increasing values of N, the first value N_1 of N which causes the resonance of synchronous modes for fairly large values of a, leads also to the coupling of modes characterized by an angular frequency equal to 3/2; therefore the ultraharmonic modes with frequency 3/2 will be included in the analysis. Hence let us consider the time development of the following perturbation fixing, for the reasons explained above, $O(\epsilon) = O(a)$:

$$\phi_{10} = \sum_{m=1}^{3} A_{\pm m}(\tau,\zeta)\psi_m^{(1)}(x,z)\exp\left[\mathrm{i}\left(k_m^{(1)}y \pm \frac{m}{2}t\right)\right] + \sum_{m=2}^{3} B_{\pm m}(\tau,\zeta)\psi_m^{(2)}(x,z)\exp\left[\mathrm{i}\left(k_m^{(2)}y \pm \frac{m}{2}t\right)\right] + \text{c.c.} \tag{4.8}$$

where the functions $\psi_m^{(1)}, \psi_m^{(2)}$ are defined by (4.2) with k equal to $k_m^{(1)}$ and $k_m^{(2)}$ respectively.

In (4.8) $A_{\pm m}, B_{\pm m}$ denote the amplitudes of the edge waves characterized by an angular frequency $m/2$ and the + and − signs indicate the negative and positive directions of edge wave propagation respectively. It is further assumed that $A_{\pm m}, B_{\pm m}$ depend on a slow temporal scale $\tau = at$ and on the slow spatial scale $\zeta = ay$. The slow temporal scale τ is introduced in order to describe the slow growth of edge wave amplitudes due to the $O(a)$ energy transfer from the basic standing wave. The interaction of edge waves characterized by wavenumbers satisfying (4.7), where a small mismatch is present, leads to slow modulations in the longshore direction which require the introduction of the spatial scale ζ. Note that although it is possible to introduce the slow variable ax as well, this is unnecessary and no greater generality is obtained.

At order a equations (2.1)–(2.4) are identically satisfied by expansion (4.8).

At order a^2 equations (2.1) and (2.2) give

$$\frac{\partial^2(\phi_{01}+\phi_{11}+\phi_{20})}{\partial x^2} + \frac{\partial^2(\phi_{01}+\phi_{11}+\phi_{20})}{\partial y^2} + \frac{\partial^2(\phi_{01}+\phi_{11}+\phi_{20})}{\partial z^2} = -2\frac{\partial^2\phi_{10}}{\partial\zeta\partial y}, \tag{4.9}$$

$$\sin\beta\frac{\partial(\phi_{01}+\phi_{11}+\phi_{20})}{\partial x} + \cos\beta\frac{\partial(\phi_{01}+\phi_{11}+\phi_{20})}{\partial z} = 0 \qquad \text{at } z = -xtg\beta, \tag{4.10}$$

while the boundary condition (2.3) combined with (2.4) yields

$$\frac{\partial(\phi_{01}+\phi_{11}+\phi_{20})}{\partial z} + \frac{\partial^2(\phi_{01}+\phi_{11}+\phi_{20})}{\partial t^2} = R(x,y,t)$$

$$= -2\frac{\partial^2\phi_{10}}{\partial t\partial\tau} + \sum_{\ell=0}^{1}\sum_{p=0}^{1}\left\{-\frac{\partial}{\partial t}\left[\frac{\partial\phi_{\ell 0}}{\partial x}\frac{\partial\phi_{p0}}{\partial x} + \frac{\partial\phi_{\ell 0}}{\partial y}\frac{\partial\phi_{p0}}{\partial y} + \frac{\partial\phi_{\ell 0}}{\partial z}\frac{\partial\phi_{p0}}{\partial z}\right] + \frac{\partial^2\phi_{\ell 0}}{\partial z^2}\frac{\partial\phi_{p0}}{\partial t} + \frac{\partial^3\phi_{\ell 0}}{\partial z\partial t^2}\frac{\partial\phi_{p0}}{\partial t}\right\} \qquad \text{at } z = 0. \tag{4.11}$$

The forcing terms on the right-hand sides of (4.9) and (4.11) suggest for $(\phi_{01}+\phi_{11}+\phi_{20})$ a temporal and longshore dependence of the form

$$E_{\pm r}^{(q)} \;;\; E_{\pm r}^{(q)}e^{+it} \;;\; E_{\pm r}^{(q)}e^{-it} \;;\; E_{\pm r}^{(q)}E_{\pm s}^{(t)} \;;\; E_{\pm r}^{(q)}\bar{E}_{\pm s}^{(t)} \;;\; \text{c.c.} \tag{4.12}$$

plus steady and ultraharmonic contributions which do not depend on y. In (4.12), $E_{\pm m}^{(j)}$ is equal to $\exp\left[\mathrm{i}\left(k_m^{(j)}y \pm (m/2)t\right)\right]$ and the indexes q,r,s,t should be allowed to vary in the ranges ($q = 1,2; t = 1,2; r = 1,2,3; s = 1,2,3$; moreover $k_1^{(j)} = k_1$ and $\mu_1^{(j)} = 0$ with $j = 1,2$).

Since $k_m^{(j)} = mk_l + a\mu_m^{(j)}$ $(m = 2, 3; j = 1, 2)$ and because of the dispersion relation (4.4), a solvability condition is required for the problem (4.9)–(4.11). Indeed since $\psi_m^{(j)} E_{\pm m}^{(j)}$ are solutions of the homogeneous problem (2.1)–(2.4), there is a bounded solution for $(\phi_{01} + \phi_{11} + \phi_{20})$ if and only if the following orthogonality conditions are satisfied:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-xtg\beta}^{0} 2\psi_m^{(j)} E_{\pm m}^{(j)} \frac{\partial \phi_{10}}{\partial y \partial \zeta} \, dz \, dx \, dy + \int_{-\infty}^{+\infty} \int_{0}^{\infty} \left[\psi_m^{(j)} E_{\pm m}^{(j)} R\right]_{z=0} dx \, dy = 0, \quad (4.13)$$

$$m = 1, 2, 3 \text{ if } j = 1, \; m = 1, 2 \text{ if } j = 2,$$

where the function R is defined by (4.11).

These conditions are similar to the well-known Fredholm alternative theorem for the simpler eigenvalue problem of ordinary differential equations. A derivation of (4.13) is given in Minzoni & Whitham (1977). This procedure leads to a system of partial differential equations which governs the development of the amplitude functions $A_{\pm m}(\tau, \zeta)$ and $B_{\pm m}(\tau, \zeta)$ in the slow temporal and spatial scales:

$$\begin{aligned} \frac{\partial A_{\pm 1}}{\partial \tau} \mp 2k_1 u_1 \frac{\partial A_{\pm 1}}{\partial \zeta} = {} & a_1 A_{\mp 1} + b_1 \bar{A}_{\pm 1} A_{\pm 2} \exp\left[i\zeta \mu_2^{(1)}\right] \\ & + c_1 \bar{A}_{\pm 2} A_{\pm 3} \exp\left[i\zeta(\mu_3^{(1)} - \mu_2^{(1)})\right] + d_1 \bar{A}_{\pm 1} B_{\pm 2} \exp\left[i\zeta \mu_2^{(2)}\right] \\ & + e_1 \bar{A}_{\pm 2} B_{\pm 3} \exp\left[i\zeta(\mu_3^{(2)} - \mu_2^{(1)})\right] + f_1 A_{\pm 3} \bar{B}_{\pm 2} \exp\left[i\zeta(\mu_3^{(1)} - \mu_2^{(2)})\right] \\ & + g_1 \bar{B}_{\pm 2} B_{\pm 3} \exp\left[i\zeta(\mu_3^{(2)} - \mu_2^{(2)})\right], \end{aligned} \quad (4.14)$$

$$\begin{aligned} \frac{\partial A_{\pm 2}}{\partial \tau} \mp k_2^{(1)} u_2^{(1)} \frac{\partial A_{\pm 2}}{\partial \zeta} = {} & b_2 A_{\pm 1}^2 \exp\left[-i\zeta \mu_2^{(1)}\right] + c_2 \bar{A}_{\pm 1} A_{\pm 3} \exp\left[i\zeta(\mu_3^{(1)} - \mu_2^{(1)})\right] \\ & + d_2 \bar{A}_{\pm 1} B_{\pm 3} \exp\left[i\zeta(\mu_3^{(2)} - \mu_2^{(1)})\right], \end{aligned} \quad (4.15)$$

$$\begin{aligned} \frac{\partial B_{\pm 2}}{\partial \tau} \mp k_2^{(2)} u_2^{(2)} \frac{\partial B_{\pm 2}}{\partial \zeta} = {} & b_3 A_{\pm 1}^2 \exp\left[-i\zeta \mu_2^{(2)}\right] + c_3 \bar{A}_{\pm 1} A_{\pm 3} \exp\left[i\zeta(\mu_3^{(1)} - \mu_2^{(2)})\right] \\ & + d_3 \bar{A}_{\pm 1} B_{\pm 3} \exp\left[i\zeta(\mu_3^{(2)} - \mu_2^{(2)})\right], \end{aligned} \quad (4.16)$$

$$\begin{aligned} \frac{\partial A_{\pm 3}}{\partial \tau} \mp \frac{2}{3} k_3^{(1)} u_3^{(1)} \frac{\partial A_{\pm 3}}{\partial \zeta} = {} & b_4 A_{\pm 1} A_{\pm 2} \exp\left[i\zeta(\mu_2^{(1)} - \mu_3^{(1)})\right] \\ & + c_4 A_{\pm 1} B_{\pm 2} \exp\left[i\zeta(\mu_2^{(2)} - \mu_3^{(1)})\right], \end{aligned} \quad (4.17)$$

$$\begin{aligned} \frac{\partial B_{\pm 3}}{\partial \tau} \mp \frac{2}{3} k_3^{(2)} u_3^{(2)} \frac{\partial B_{\pm 3}}{\partial \zeta} = {} & b_5 A_{\pm 1} A_{\pm 2} \exp[i\zeta(\mu_2^{(1)} - \mu_3^{(2)})] \\ & + c_5 A_{\pm 1} B_{\pm 2} \exp\left[i\zeta(\mu_2^{(2)} - \mu_3^{(2)})\right]. \end{aligned} \quad (4.18)$$

The expressions which give the coefficients as functions of the parameters of the problem are very lengthy and their derivation is tedious but straighforward. For the sake of brevity, they are given in the Appendix. The values of $u_m^{(1)}$ and $u_m^{(2)}$ are related to the group velocity of the free modes, the coefficient a_1 describes the energy transfer from the incoming wave to the subharmonic modes, while all the other coefficients result from the nonlinear interactions. It is worth pointing out that all the coefficients turn out to be real.

Obvious simplifications in the system (4.14)–(4.18) are present when some of the modes considered in (4.8) are not resonating.

The solution of (4.14)–(4.18) describes the slow time development and longshore modulation of the amplitudes of the edge waves (4.8) which exchange energy among each other through nonlinear interactions and extract energy from the incident wave field through the intermediary of the subharmonic modes.

The ζ-dependence of the coefficients of the partial differential equations (4.14)–(4.18) can be removed simply by defining the new variables $\tilde{A}_{\pm m} = A_{\pm m} \exp[\mu_m^{(1)}\zeta]$ and $\tilde{B}_{\pm m} = B_{\pm m} \exp[\mu_m^{(2)}\zeta]$ with $m = 2, 3$. It is thus possible to look for special solutions of $\tilde{A}_{\pm m}, \tilde{B}_{\pm m}$ which do not depend on ζ. By focusing on the special case of amplitudes $\tilde{A}_{\pm}, \tilde{B}_{\pm}$ which do not depend on ζ, a system of amplitude equations is found where $\tilde{A}_{\pm m}, \tilde{B}_{\pm m}$ replace $A_{\pm m}, B_{\pm m}$, the coefficients are constants and the partial derivatives with respect to ζ vanish. This is the system of amplitude equations, the solutions of which is described in the following.

5. Discussion of the results

Before presenting the results, it is necessary to briefly discuss the relevance of the theory and its range of applicability. The problem is characterized by the presence of the parameter N which is equivalent to the beach slope β ($\beta = \pi/(2N)$). As previously pointed out, if sufficiently large values of N are considered, the couplings described above are possible or not depending on the value of a, μ_m being a parameter of order one. However, as in all perturbation approaches, it is not possible to quantify exactly the limiting values of a and the results to be described should be interpreted as giving an indication of the qualitative behaviour of the solution in different possible regimes. For example the dispersion relation (4.4) shows that for $N = 10$, the coupling between the subharmonic modes with $n = 1$ and the synchronous modes with $n = 3$ takes place if fairly large values of a but smaller than one are considered. Moreover when increasing values of a are considered resonance is affected by the 3/2-frequency components with $n = 4$. Hence for this particular value of N three broad regimes (denoted Regimes I, II and III respectively) can be identified when increasing values of a are considered.

(1) Regime I: no coupling is possible and the time development of $A_{\pm 1}$ turns out to be that described by Minzoni & Whitham (1977), while the synchronous and 3/2-frequency edge waves neither grow nor amplify.

(2) Regime II: the subharmonic–synchronous edge wave interaction takes place and the solution of the system (4.14)–(4.18) shows that both the subharmonic and synchronous edge waves grow.

(3) Regime III: the interaction mentioned above is affected by the 3/2-frequency components and by other synchronous edge waves. In this case the synchronous modes are damped and there is an energy transfer from the subharmonic components to the 3/2-frequency edge waves which consequently grow. Moreover other couplings are possible as described in more detail in the following.

It should be pointed out that from a practical point of view only modes which satisfy condition (4.7) with a maximum relative error of 50% have been considered.

The nonlinear complex system of ordinary differential equations (4.14)–(4.18) is solved numerically here. A fourth-order Runge–Kutta scheme is used to obtain computational results for different beach slope β and different initial conditions. Representative numerical solutions are presented in the following. In order for the theory to be meaningful, limits should be forced to the amplitude functions. Indeed for the expansion (4.8) to be rational the quantities $a\tilde{A}_{\pm m}$ and $a\tilde{B}_{\pm m}$ should be much smaller than one. For this reason the numerical integration has been stopped as soon

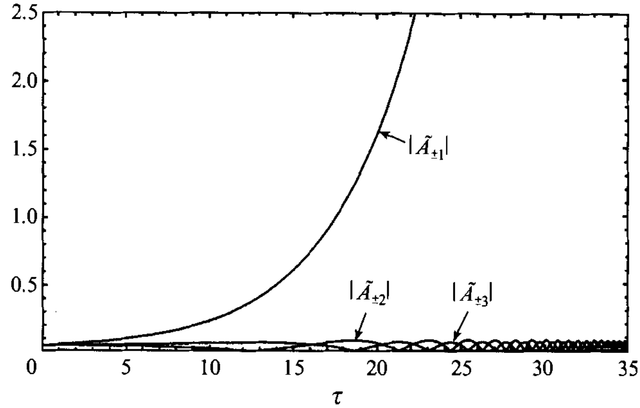

FIGURE 1. Time development of the modulus of the amplitudes $\tilde{A}_{\pm m}(m = 1,2,3)$ for $N = 4$ and $a = 0.1$. For the subharmonic, synchronous and 3/2-frequency edge waves, the modes with $n = 0,1,1$ (see (4.4)) respectively are considered. The initial values of the amplitudes are $\tilde{A}_{\pm m}(0) = (0.05, 0)$.

as $a\tilde{A}_{\pm m}$ or $a\tilde{B}_{\pm m}$ exceeds values of order one. Indeed only a fully nonlinear solution can give indications about the time behaviour of the solution for large amplitudes. Since qualitatively different behaviours of the solution are found on varying the slope of the beach β, in the following the results will be described for each value of N.

$N < 4$

No resonance is possible for values of N smaller than 4, indeed in this case neither the condition $k_2/2 \approx k_1$ nor the condition $k_3/3 \approx k_1$ are satisfied even when fairly large values of a are considered. For such values of N, the theory of Minzoni & Whitham (1977) applies.

$N = 4, 5$

For $N = 4$, the coupling among subharmonic modes with $n = 0$ in (4.4), the synchronous and 3/2-frequency modes both with $n = 1$ becomes possible but only for fairly large values of a. In this case equations (4.16) and (4.18) disappear from the system (4.14)–(4.18). Because of the sensitivity of the results to the initial conditions, an exhaustive numerical investigation has been performed varying $\tilde{A}_{\pm m}(0)$ even though attention has been focused on small values, since at the initial stages edge waves are only small perturbations of the incoming wave field. In all cases the numerical results show that no energy transfer takes place from the subharmonic to the other edge waves. An example of the results is shown in figure 1. Hence the analysis leads to the conclusion that the exponential growth of the subharmonic modes predicted by Minzoni & Whitham (1977) (see figure 2) is almost unaffected by the nonlinear interaction with the other modes. Similar results are obtained for $N = 5$.

$N = 6, 7$

Table 1 shows the wavenumbers k_1, k_2, k_3 for the three possible values of n and for $\sigma = 1/2, 1, 3/2$ and $N = 6$. Two different couplings are possible even though they take place for fairly large values of a but smaller than one. Case 1: the two subharmonic edge waves with $n = 0$, the synchronous edges with $n = 1$ and the pair with $n = 2$, the four 3/2-frequency edge waves with $n = 1$ and $n = 2$. Indeed the relative distance of the wavenumbers of the synchronous edge waves with both

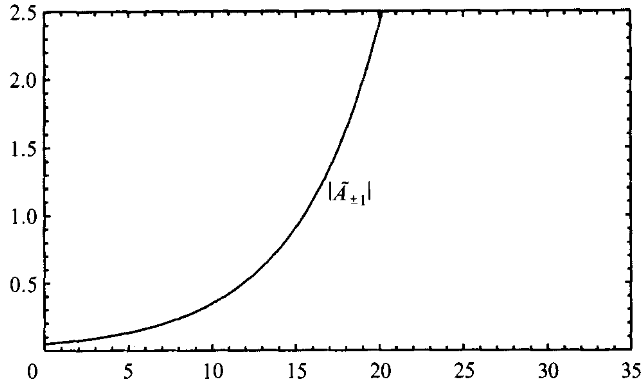

FIGURE 2. Time development of the modulus of the amplitudes $\tilde{A}_{\pm 1}$ for $N = 4$ and $a = 0.1$. The nonlinear couplings are neglected and for the subharmonic edge waves the mode with $n = 0$ (see (4.4)) is considered. The initial values of the amplitudes are $\tilde{A}_{\pm 1}(0) = (0.05, 0)$.

$\sigma_n =$	1/2	1	3/2
n			
0	0.97	3.86	8.69
1	0.35	1.41	3.18
2	0.26	1.04	2.33

TABLE 1. Wavenumber k for different values of n, σ_n and for $N = 6$.

$n = 1$ and $n = 2$ from twice the wavenumber k_1 of the subharmonic edge waves with $n = 0$ turns out to be $|k_2^{(1)} - 2k_1|/2k_1 = 0.27$ and $|k_2^{(2)} - 2k_1|/2k_1 = 0.46$ and the relative distance of the wavenumbers of the 3/2-frequency edge waves with $n = 1,2$ from $3k_1$ is $|k_2^{(1)} - 3k_1|/3k_1 = 0.09$ and $|k_3^{(2)} - 3k_1|/3k_1 = 0.20$ respectively. Case 2: the two subharmonic edge waves with $n = 1$ and the synchronous pair with $n = 2$ ($|k_2^{(1)} - 2k_1|/2k_1 = 0.49$). In case 1 results similar to those previously described are obtained, i.e. only the subharmonic edge waves increase their amplitude. It should be pointed out that in this case the time behaviour of the components described by (4.8) is affected by the edge waves characterized by σ equal to 2 and 5/2 with $n = 2$ in (4.4). By considering the index m appearing in (4.8) ranging up to 5, it has been verified that the inclusion of the components characterized by $\sigma = 2$, 5/2 does not modify the qualitative behaviour of the results previously described. The details are not described for brevity. The resonance of the modes characterized by $\sigma = 2$, 5/2, ... is also present for larger values of N but only when the subharmonic modes with $n = 0$ in (4.4) are considered. Also in these cases it has been verified that their presence does not modify the results obtained on the basis of (4.8).

When case 2 is considered, qualitatively and quantitatively different results are obtained with respect to the dependence of the magnitude and phase of the edge wave amplitudes. If propagating edge waves are considered (i.e. $\tilde{A}_{+m}(0) \neq \overline{\tilde{A}}_{-m}(0)$), an energy transfer from the subharmonic modes to the synchronous ones takes place which causes the rapid growth of the latter modes (see figure 3). When steady edge waves are considered or equivalently when the initial values of the amplitudes are such that $\tilde{A}_{+m}(0) = \overline{\tilde{A}}_{-m}(0)$, a periodic behaviour is found (see figure 4). In

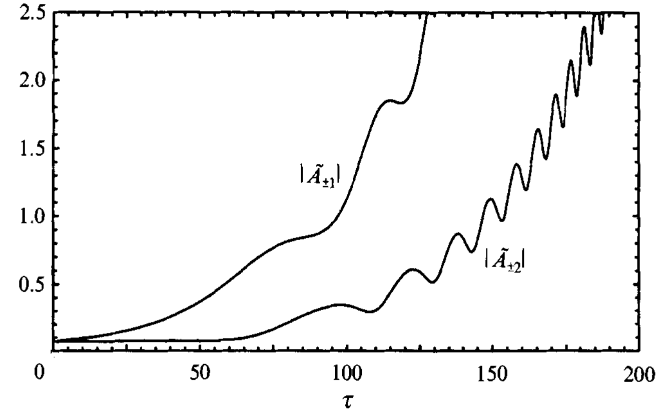

FIGURE 3. Time development of the modulus of the amplitudes $\tilde{A}_{\pm m}(m = 1, 2)$ for $N = 6$ and $a = 0.1$. For the subharmonic and synchronous edge waves, the modes with $n = 1,2$ (see (4.4)) respectively are considered. The initial values of the amplitudes are $\tilde{A}_{\pm m}(0) = (0.05, 0.05)$.

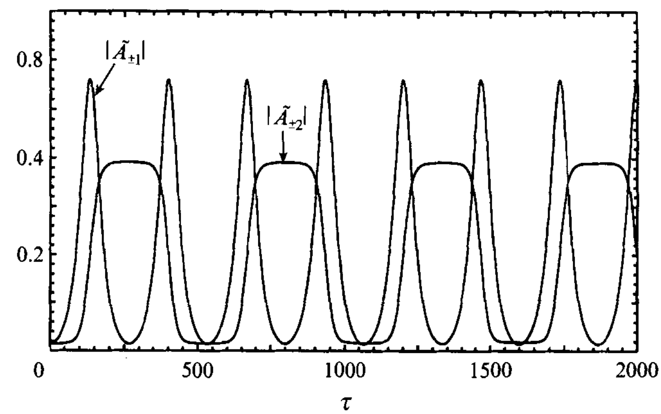

FIGURE 4. As figure 3 but the initial values of the amplitudes are $\tilde{A}_{+m}(0) = (0.01, 0.01)$, $\tilde{A}_{-m}(0) = (0.01, -0.01)$.

this case the solution attains a regime configuration with $\tilde{A}_{\pm 2}$ of order one. If the solution is obtained for a longer time, modulations of the amplitude functions are present. However, such modulations are characterized by a quite long period and call for a further slow time scale to be properly described. It is worth pointing out that when the conditions $\tilde{A}_{+m}(0) = \overline{\tilde{A}}_{-m}(0)$ are not exactly satisfied, the small perturbations superimposed on the initially steady edge waves amplify and lead to the explosion of both the subharmonic and synchronous modes (see figure 5). Finally, qualitatively different solutions are found when $\tilde{A}_{\pm m}(0)$ are all real. In this case the amplitude functions attain a steady configuration with vanishing subharmonic modes and synchronous modes of amplitude order one (see figure 6). However, this steady solution is unstable and it is found because there is no numerical mechanism triggering the imaginary parts of $\tilde{A}_{\pm m}$. When small imaginary contributions are given to $\tilde{A}_{\pm m}(0)$ the solution shifts towards those previously described, i.e. both $A_{\pm 1}$ and $A_{\pm 2}$ show an explosive behaviour.

These numerical findings are supported by an analysis of the fixed points of the system (4.14)–(4.18). Indeed when only two subharmonic and two synchronous modes propagating in opposite directions are considered, the fixed points of the

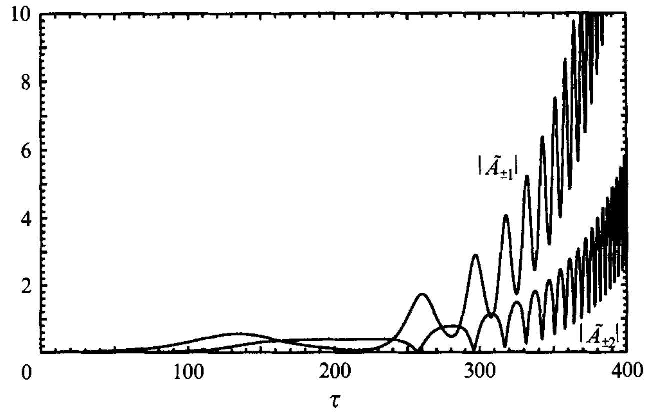

FIGURE 5. As figure 3 but the initial values of the amplitudes are $\tilde{A}_{+1}(0) = (0.01, 0.01)$, $\tilde{A}_{-1}(0) = (0.010001, -0.010001), \tilde{A}_{+2}(0) = (0.01, 0.01)$, $\tilde{A}_{-2}(0) = (0.009999, -0.00999)$.

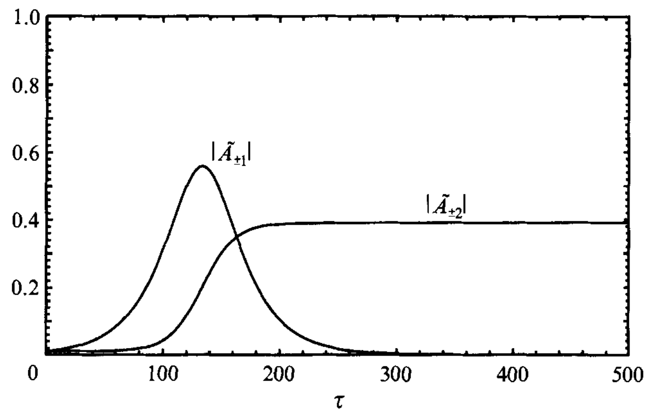

FIGURE 6. As figure 3 but the initial values of the amplitudes are $\tilde{A}_{\pm m}(0) = (0.01, 0)$.

system (4.14)–(4.18) can be easily evaluated. It turns out that

$$\tilde{A}^{(f)}_{\pm 1} \equiv 0; \quad \tilde{A}^{(f)}_{+2}, \tilde{A}^{(f)}_{-2} = \text{const.} \tag{5.1}$$

where the superscript f denotes equilibrium conditions. With a proper choice of the origin of the y-axis, $\tilde{A}^{(f)}_{+2}$ can be made real. In order to study the stability of these fixed points, the nonlinear system (4.14)–(4.18) can been linearized in the neighbourhoods of $\tilde{A}^{(f)}_{\pm 1}$ and $\tilde{A}^{(f)}_{\pm 2}$ and the time derivative of the amplitudes of small perturbations superimposed on the steady states are thus proportional to the perturbations themselves through the Jacobian matrix of the system (4.14)–(4.18). The Routh–Hurwitz criterion applied to the polynomial which provides the eigenvalues of the Jacobian matrix shows that the fixed points are unstable. Indeed four eigenvalues turn out to be equal to zero, two are negative but the last two are positive.

For a beach slope $\beta = \pi/14$, i.e. for $N = 7$, the results are similar to those found for $N = 6$.

$N = 8, 9$

For $N = 8$ and $N = 9$, the results are qualitatively similar to those described for $N = 6$ and $N = 7$ but for the synchronous edge waves with $n = 3$ replacing those with $n = 2$.

σ_n =	1/2	1	3/2
n			
0	1.60	6.39	14.38
1	0.55	2.20	4.96
2	0.35	1.41	3.18
3	0.28	1.12	2.53
4	0.25	1.01	2.28

TABLE 2. Wavenumber k for different values of n, σ_n and for $N = 10$.

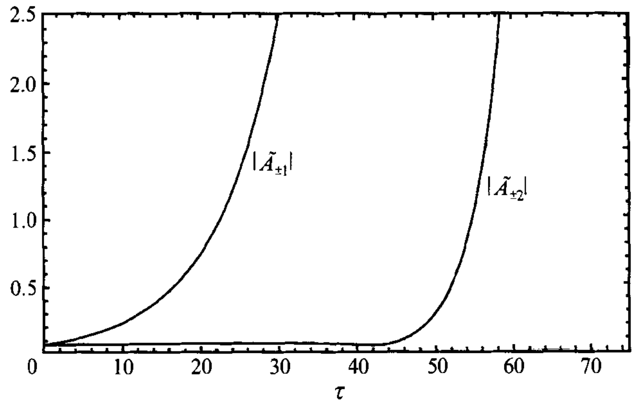

FIGURE 7. Time development of the modulus of the amplitudes $\tilde{A}_{\pm m}(m = 1, 2)$ for $N = 10$ and $a = 0.1$. For the subharmonic and synchronous edge waves, the modes with $n = 1,3$ (see (4.4)) respectively are considered. The initial values of the amplitudes are $\tilde{A}_{\pm m}(0) = (0.05, 0.05)$.

$N = 10$

A richer range of results is obtained for $N = 10$. Table 2 shows the wavenumbers k_1, k_2, k_3 for $\sigma = 1/2, 1, 3/2$ respectively and for the five possible values of n and allows the relative distance of $k_m^{(j)}$ from $mk_1 (j = 1, 2;\ m = 2, 3)$ to be evaluated.

As previously pointed out, the time development of the different edge wave modes depends on the amplitude of the incoming wave. When a is quite small, the nonlinear interactions described in the present work are absent, since conditions (4.7) cannot be satisfied with μ_m of order one. On increasing a, the coupling among the subharmonic edge waves with $n = 1$ and the synchronous ones with $n = 3$ becomes possible. An example of the results is plotted in figure 7 and shows that an energy transfer from the subharmonic modes to the synchronous ones is present but is weak and negative at the beginning. Hence the synchronous modes after an initial decay significantly grow only when the subharmonic ones attain quite large amplitudes and the analysis has no longer any physical meaning. A further increase of a leads to different situations. Case 1: the coupling just described is affected by the presence of a second pair of synchronous modes with $n = 4$ in (4.4) and by the 3/2-frequency modes with $n = 4$. Also in this case the growth of the synchronous modes is not triggered. Case 2: the subharmonic edge waves with $n = 0$, the synchronous edge waves with $n = 1$ and two pairs of 3/2-frequency edge waves with $n = 1$ and $n = 2$ respectively interact. This case gives results similar to those obtained for the same coupling but for smaller N (see figure 8). Case 3: the subharmonic edge waves with $n = 2$ and the synchronous edge waves with $n = 4$ resonate. This case gives further support to the conclusion that

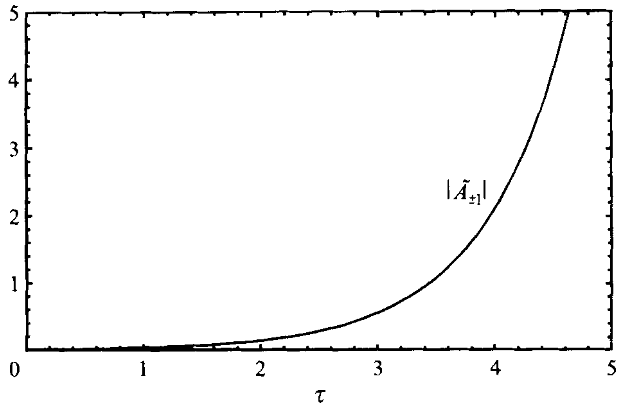

FIGURE 8. Time development of the modulus of the amplitudes $\tilde{A}_{\pm m}(m = 1, 2)$ for $N = 10$ and $a = 0.1$. For the subharmonic, synchronous and 3/2-frequency edge waves, the modes with $n = 0,1,1$ and 2 (see (4.4)) respectively are considered. The initial values of the amplitudes are $\tilde{A}_{\pm m}(0) = (0.01, 0)$.

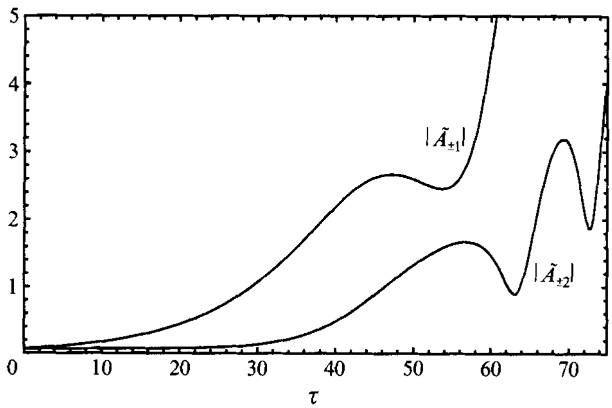

FIGURE 9. Time development of the modulus of the amplitudes $\tilde{A}_{\pm m}(m = 1, 2)$ for $N = 10$ and $a = 0.1$. For the subharmonic and synchronous edge waves, the modes with $n = 2,4$ (see (4.4)) respectively are considered. The initial values of the amplitudes are $\tilde{A}_{\pm m}(0) = (0.05, 0.05)$.

the interaction of subharmonic modes with synchronous ones without the presence of 3/2-frequency components may lead to the growth of edge waves with the same frequency as the incoming wave. Indeed the results plotted in figure 9 show a rapid grow of $\tilde{A}_{\pm 2}$. Also in this case a regime configuration can be found when the initial values are such that $\tilde{A}_{-m} = \overline{\tilde{A}_{+m}}$ (see figure 10).

$N > 10$

Of course on increasing N, i.e. considering smaller values of β, the complexity of the problem increases and the expansion (4.8) does not contain enough modes to be able to take into account all the possible interactions which may take place for relatively large values of a. Only if a is extremely small is the system (4.14)–(4.18) still adequate to describe the nonlinear interactions of different modes. For example when $N = 30$ and a assumes values of about 10^{-2} the following couplings are possible: case 1, the subharmonic modes with $n = 5$, the synchronous modes with $n = 5$ and the 3/2-frequency modes with $n = 8$; case 2, the subharmonic modes with $n = 3$

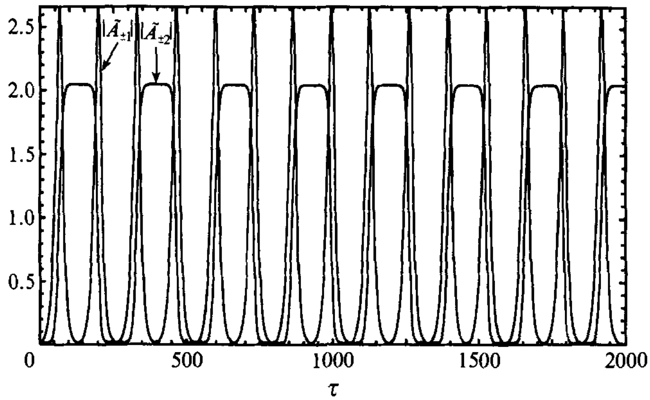

FIGURE 10. As figure 9 but the initial values of the amplitudes are $\tilde{A}_{+m}(0) = (0.01, 0.01)$, $\tilde{A}_{-m}(0) = (0.01, -0.01)$.

and the synchronous modes with $n = 7$; case 3, the subharmonic modes with $n = 4$ and the synchronous modes with $n = 10$ and 11. However, when increasing values of a are considered, there are many more synchronous and 3/2-frequency modes all interacting with a given subharmonic mode. A full nonlinear calculation is called for at this stage.

6. Conclusions

A nonlinear mechanism has been outlined which can trigger the growth of synchronous edge waves on a plane reflective beach subject to a normally incident monochromatic wave.

Indeed values of the beach slope and of the incoming wave amplitude exist such that an energy transfer takes place from the incoming wave to the synchronous edge waves through the intermerdiary of subharmonic components.

This mechanism may coexist with that described by Rockliff (1978) which however causes the growth of synchronous edge waves on a temporal scale much longer than the present one. Moreover the equilibrium amplitude of the synchronous edge waves considered by Rockliff (1978) is much smaller than that of the subharmonic ones which are present for the same conditions. On the contrary, even though the system (4.14)–(4.18) does not lead to any equilibrium configuration (except for some particular cases), the theory suggests that the considered subharmonic and synchronous edge waves are characterized by amplitudes with the same order of magnitude which usually is larger than that of the incoming wave. Hence a theoretical explanation of the experimental observations by Galvin (1965), Guza & Inman (1975) and other authors is provided. An order of magnitude analysis of the problem (2.1)–(2.4) reveals that the regime configuration, which is attained by subharmonic and synchronous edge waves growing because of their mutual interaction, cannot be found by means of a weakly nonlinear analysis but only by resorting to a numerical solution of the full problem.

Moreover the results seem to indicate that for small beach slope the nonlinear interactions become very complicated. Numerical investigations of the phenomenon as well as further experimental observations can provide further insight into the actual process occurring in the latter regime.

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Appendix

$$u_n^{(m)} = \int_{-xtg\beta}^{0} \int_0^{\infty} (\psi_n^{(m)})^2 \,\mathrm{d}x\,\mathrm{d}z \Big/ \int_0^{\infty} (\psi_n^{(m)})^2_{z=0} \mathrm{d}x.$$

$$a_1 = \int_0^{\infty} \psi_1^{(1)} \left\{ -\left[\frac{\partial \psi_1^{(1)}}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial \psi_1^{(1)}}{\partial z}\frac{\partial G}{\partial z} \right] + \left[\frac{\partial^2 \psi_1^{(1)}}{\partial z^2} G - \tfrac{1}{2}\psi_1^{(1)} \left(\frac{\partial^2 G}{\partial z^2} - \frac{\partial G}{\partial z} \right) - \tfrac{1}{4}\frac{\partial \psi_1^{(1)}}{\partial z} G \right] \right\}_{z=0} \mathrm{d}z \Big/ \int_0^{\infty} (\psi_1^{(1)})^2_{z=0} \mathrm{d}x;$$

$$b_1 = \mathscr{F}_+^{(1,1,1)}(1,1,2,1); \quad c_1 = \mathscr{F}_+^{(1,1,1)}(1,2,3,1); \quad d_1 = \mathscr{F}_+^{(1,1,2)}(1,1,2,1),$$

$$e_1 = \mathscr{F}_+^{(1,1,2)}(1,2,3,1); \quad f_1 = \mathscr{F}_+^{(1,2,1)}(1,2,3,1); \quad g_1 = \mathscr{F}^{(1,2,2)}(1,2,3,1),$$

$$b_2 = \mathscr{F}_-^{(1,1,1)}(2,1,1,0); \quad c_2 = \mathscr{F}_+^{(1,1,1)}(2,1,3,1); \quad d_2 = \mathscr{F}_+^{(1,1,2)}(2,1,3,1),$$

$$b_3 = \mathscr{F}_-^{(2,1,1)}(2,1,1,0); \quad c_3 = \mathscr{F}_+^{(2,1,1)}(2,1,3,1); \quad d_3 = \mathscr{F}_+^{(2,1,1)}(2,1,3,1),$$

$$b_4 = \mathscr{F}_-^{(1,1,1)}(3,1,2,1); \quad c_4 = \mathscr{F}_-^{(1,1,2)}(3,1,2,1),$$

$$b_5 = \mathscr{F}_-^{(2,1,1)}(3,1,2,1); \quad c_5 = \mathscr{F}_-^{(2,1,2)}(3,1,2,1),$$

where

$$\mathscr{F}_\pm^{(j,l,m)}(n,p,q,\lambda) = \int_0^{\infty} \psi_n^{(j)} \left\{ -\frac{n}{(2-\lambda)} \left[\frac{\partial \psi_p^{(l)}}{\partial x}\frac{\partial \psi_q^{(m)}}{\partial x} \pm k_p^{(l)} k_q^{(m)} \psi_p^{(l)} \psi_q^{(m)} + \frac{\partial \psi_p^{(l)}}{\partial z}\frac{\partial \psi_q^{(m)}}{\partial z} \right] + \left[\frac{q}{2}\frac{\partial^2 \psi_p^{(l)}}{\partial z^2} \psi_q^{(m)} - \left(\frac{p}{2}\right)^2 \frac{q}{2}\frac{\partial \psi_p^{(l)}}{\partial z} \psi_q^{(m)} \pm \lambda \left(\frac{p}{2}\left(\frac{q}{2}\right)^2 \psi_q^{(l)} \frac{\partial \psi_q^{(m)}}{\partial z} - \frac{p}{2}\psi_p^{(l)} \frac{\partial^2 \psi_q^{(m)}}{\partial z^2} \right) \right] \right\}_{z=0} \mathrm{d}x \Big/ \int_0^{\infty} n(\psi_n^{(j)})^2_{z=0} \mathrm{d}x$$

and

$$G(x,z) = \sum_{m=1}^{N} \frac{\mathscr{A}_m}{4N^{1/2}} \exp[\mathscr{B}_m(x + \mathrm{i}z)] + \mathrm{c.c.}$$

REFERENCES

GALVIN, C. J. 1965 Resonant edge waves on laboratory beaches. *Eos Trans. AGU* **46**, 122.

GUZA, R. T. & BOWEN, A. J. 1976 Finite amplitude edge waves. *J. Mar. Res.* **34**, 269–293.

GUZA, R. T. & DAVIS, R. E. 1974 Excitation of edge waves by waves incident on a beach. *J. Geophys. Res.* **79**, 1285–1291.

GUZA, R. T. & INMAN, D. L. 1975 Edge waves and beach cusps. *J. Geophys. Res.* **80**, 2997–3011.

MILES, J. W. 1990. Parametrically excited standing edge waves. *J. Fluid Mech.* **214**, 43–57.

MILES, J. W. & HENDERSON, D. 1990 Parametrically forced surface waves. *Ann. Rev. Fluid Mech.* **22**, 143–146.

MINZONI, A. A. 1976 Nonlinear edge waves and shallow-water theory. *J. Fluid Mech.* **74**, 369–374.

MINZONI, A. A. & WHITHAM, G. B. 1977 On the excitation of edge waves on beaches. *J. Fluid Mech.* **79**, 273–287.

ROCKLIFF, N. 1978 Finite amplitude effect in free and forced edge waves. *Math. Proc. Camb. Phil. Soc.* **83**, 463–479.

SIMMONS, W. F. 1969 A variational method for weak resonant wave interactions. *Proc. R. Soc. Lond.* A **309**, 551–579.

STOKER, J. J. 1957 *Water Waves.* Interscience.

STOKES, G. G. 1846 Report on recent researchers in hydrodynamics. *Brit. Assoc. Rep.* Part 1, 1.

URSELL, F. 1952 Edge waves on a sloping beach. *Proc. R. Soc. Lond.* A **214**, 79–97.

WERNER, B. T. & FINK, T. M. 1993 Beach cusps as self-organized patterns. *Science* **260**, 968–971.

WHITHAM, G. B. 1976 Nonlinear effects in edge waves. *J. Fluid Mech.* **74**, 353–368.